Oscillating Delta Wings with Attached Shock Waves

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An unsteady flow theory is presented for studying the flowfield in the compression side of an oscillating flat delta wing with attached shock waves. Regular perturbation methods are used to analyze the in-phase and out-of-phase flow components for small amplitudes and reduced frequencies. In particular, the out-of-phase flow is found to be "quasiconical," thus a pressure formulation can be realized. In the outboard region, where the crossflow is supersonic, exact solutions are found representing parallel surfaces of isobars. In the central region where the crossflow is subsonic, the problem is reduced to that of an ordinary-differential equation by a spanwise integration technique. Closed-form solutions are obtained for all cases. Numerical examples are presented to exhibit the dependence of the damping derivatives on several flow and geometrical parameters. Neutral damping boundaries are also given. It is found that the damping derivatives are generally less sensitive to the sweepback-angle and the freestream Mach number variations than to the mean-incidence variations, except near the shock detachment. Critical assessments, improvement schemes and future extensions were also discussed.

Nomenclature

$(\ )_\infty$	= freestream quantities
$(\tilde{\ })$	= physical quantities
H	$= \lambda \tan\phi_0$
H'	$= (1-H^2)^{1/2}$
L	= physical length of the root chart (Fig. 1a).
M_0	= aft-shock Mach number for 2d wedge flow
M_*	= aft-shock Mach number for skewed wedge flow
P_0, u_0, v_0...etc	= aft-shock properties for 2d wedge flow
P_*, u_*, v_*...etc	= aft-shock properties for skewed wedge flow
x_0	= pivot position from wing apex
α_0, β_0	= deflection angle and shock angle for a 2d wedge flow (Fig. 1c)
α_I, β_I	= deflection angle and shock angle for a skewed wedge flow (Fig. 1c)
δ	= small parameter indicating departure from 2d wedge flow (i.e. $\delta = \tan\chi$)
$\bar{\epsilon}$	= physical amplitude of oscillation
$\epsilon = \bar{\epsilon}/\alpha_0$	= nondimensional amplitude of oscillation
$\Delta(x,t)$	$= (x-x_0)e^{ik_* t}$, the perturbed surface function
Λ	$= \cot\chi$
λ_*	$= (M_*^2 - 1)^{1/2}$
λ	$= (M_0^2 - 1)^{1/2}$
$\bar{\alpha}$	$= \lambda P_* / \gamma P_0^* M_0^2$
$\bar{\beta}$	$= u_* / u_0$
ϕ_0, ϕ_I	= angles measured from the plane shock to the surface for 2d wedge flow and skewed wedge flow respectively (Fig. 1c)
χ	= sweepback angle of the delta wing (Fig. 1a)
τ_I	= angle in the side plane measured from the leading edge to the freestream direction (Fig. 1c)

Received July 30, 1976: revision received March 8, 1977.
Index categories: Nonsteady Aerodynamics; Supersonic and Hypersonic Flow; Aircraft Handling, Stability, and Control.

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I. Introduction

WITH the advent of the space shuttle and high performance military aircrafts, the design knowledge for such types of configuration is of increasing demand. In particular, the urgent need for aerodynamic load and dynamic stability predictions for these vehicles at high incidence during the course of re-entry or maneuver have been pointed out by Townend[1] and recently by Orlik-Rückemann.[2] When descending at supersonic/hypersonic speeds, the vehicles encounter shock waves which are usually strong and can be either detached or attached to the wing leading edges. To enhance the lift to drag ratio, the shock attachment case is often preferred.[1] In this case, since there is no communication between the upper surface (leeward side) and the lower surface (windward side), the aerodynamic problems can be tackled independently. Hence the flowfield of the lower surface, steady or unsteady, can be examined first. In this regard, other considerations involving vortex-sheet roll up, or various scales of flow separation caused by the high incidence, can be totally disregarded for the time being.

Generally, the shock attachment criterion depends on four parameters, namely the freestream Mach number M_∞, the sweepback angle χ, the flow incidence α_0, and the ratio of the specific heats of the gas γ. The flowfield of interest is the compression region defined as one which is bounded by the lower part of the wing surface and the enclosed shock waves (Fig. 1a). In the past, steady problems of this kind have drawn the attention of many investigators. Numerical solutions were first obtained by Fowell[3] and Babaev,[4] and more recently by Voskresenskii,[5] South and Klunker[6] and Kutler and Lomax[7] among others. Analytical approaches were first given by Malmuth.[8,9] His method, based on the framework of hypersonic small disturbance theory, is restricted to the confined case that the central region bounded by the Mach cone has to be smaller than the outboard region. Hui[10,11] later generalized Malmuth's approach in constituting a unified supersonic/hypersonic theory which removes the earlier restrictions. Also obtained by perturbation technique, Hui's solution in the central region accounts for the small departure, of $0(\delta)$, from the reference wedge flow. Here δ is a small parameter defined as $\delta = \tan\chi$, characterizing the deviation of the flow from the reference flow. This solution is then patched with the exact solution in the outboard region. Hui's theory interestingly gives almost identical results when compared with those calculated by the numerical methods. Furthermore, it should be remarked that although δ is assumed small, this assumption really does not need be restricted to the small χ cases. In fact, in the steady cases,[10,11] it turns out that good results are obtained for all χ's insofar as in each case the shock attachment to the leading edges is assured. This is expected to be true for unsteady cases also.

By contrast, there appears to be little information available about the unsteady flow of such a problem. In the two-dimensional case, on the other hand, Hui's exact theory[12] predicts domains of dynamic instability for wedges and flat plates in a certain Mach number range at high enough incidences. For the shock detachment case, Hui and Hemdan[13] recently studied the unsteady delta wing problem based on the hypersonic thin shock layer approach. Otherwise, existing methods in the past for analyzing the unsteady three-dimensional flow problems were based on the supersonic potential flow assumption,[14,17] the piston analogy,[18] or were restricted to the unsteady Newtonian theory.[19] So far, it appears that there is virtually no satisfactory method available in the intermediate Mach number range.

The purpose of the present study therefore is to develop an analytical method in which the Mach number range can be unified and the unsteady shock wave effects, hence the rotationality, is properly accounted for. We attempt to generalize Hui's steady theory[10] for oscillating delta wings in pitch. The only provision is that the shock waves must be attached to the wing leading edges at all times. Emphasis is placed on the analysis of the unsteady flow disturbances over the compression region and hence leading to the calculation of the stability derivatives.

II. Statement of the Problem

Consider a flat-bottomed delta wing of sweepback angle χ in supersonic/hypersonic flow with attached shock waves as shown in Fig. 1a. A space-fixed system based on cartesian coordinate $(\bar{x},\bar{y},\bar{z})$ is employed: the plane $\bar{y}=0$ contains the mean position of the lower surface of the wing. The origin is then fixed at the mean position of the wing apex. Figure 1a is drawn bottom up with the mean position of the lower surface $\bar{y}=0$ at an incidence α_0, while the plane $\bar{z}=0$ is the plane of symmetry. With respect to the field point $\bar{r}=(\bar{x},\bar{y},\bar{z})$, the velocity of the flowfield is defined as $\bar{V}=(\bar{u},\bar{v},\bar{w})$.

Thus, for an inviscid flow of an adiabatic gas, the mathematical problem consists of solving the following set of equations of mass, momentum, and energy, i.e.

$$\frac{\partial\bar{\rho}}{\partial\bar{t}}+\bar{\nabla}\cdot(\bar{\rho}\bar{V})=0 \qquad (1a)$$

$$\frac{\partial\bar{V}}{\partial\bar{t}}+\bar{V}\cdot\bar{\nabla}\bar{V}+\frac{\bar{\nabla}\bar{p}}{\bar{\rho}}=0 \qquad (1b)$$

$$\frac{\partial}{\partial\bar{t}}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right)+\bar{V}\cdot\bar{\nabla}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right)=0 \qquad (1c)$$

where $\bar{p}$, $\bar{\rho}$ and $\bar{t}$ are the physical quantities of pressure, density, and real time, and γ the ratio of the specific heats of the gas. The ∇ operator reads

$$\bar{\nabla}=\left(\frac{\partial}{\partial\bar{x}},\ \frac{\partial}{\partial\bar{y}},\ \frac{\partial}{\partial\bar{z}}\right)$$

The equations are subject to the general flow-tangency condition i.e.,

$$\left(\frac{\partial}{\partial\bar{t}}+\bar{V}\cdot\bar{\nabla}\right)\bar{S}=0 \quad \text{at} \quad \bar{S}(\bar{r},\bar{t})=0 \qquad (2)$$

which requires no normal velocity at the general wing surface $\bar{S}(\bar{r},\bar{t})$ at all times and to the time-dependent Rankine-Hugoniot conditions i.e.,

$$\left\langle\bar{\rho}\left(\frac{\partial\bar{G}}{\partial\bar{t}}+\bar{V}\cdot\bar{\nabla}\bar{G}\right)\right\rangle=0 \qquad (3a)$$

$$\left\langle\bar{\rho}\left(\frac{\partial\bar{G}}{\partial\bar{t}}+\bar{V}\cdot\bar{\nabla}\bar{G}\right)^2+\left(\bar{\nabla}\bar{G}\right)^2\bar{p}\right\rangle=0 \qquad (3b)$$

$$\left\langle\bar{V}\cdot\tau\right\rangle=0 \quad \text{at} \quad \bar{G}(\bar{r},\bar{t})=0 \qquad (3c)$$

$$\left\langle\bar{V}\cdot s\right\rangle=0 \qquad (3d)$$

$$\left\langle\frac{1}{2}\left(\frac{\partial\bar{G}}{\partial\bar{t}}+\bar{V}\cdot\bar{\nabla}\bar{G}\right)^2+(\bar{\nabla}\bar{G})^2\cdot\frac{\gamma}{\gamma-1}(\bar{p}/\bar{\rho})\right\rangle=0 \qquad (3e)$$

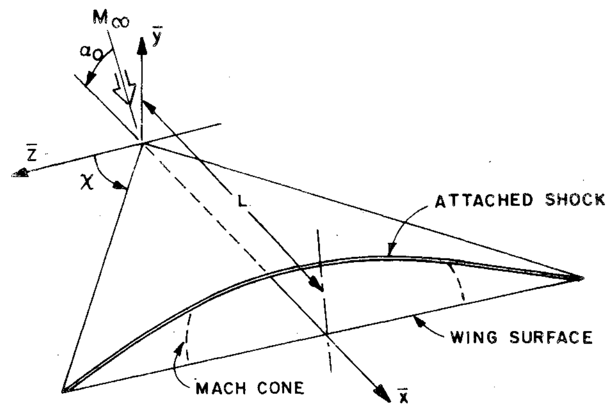

Fig. 1a Flat-bottomed delta wing placed in its mean position.

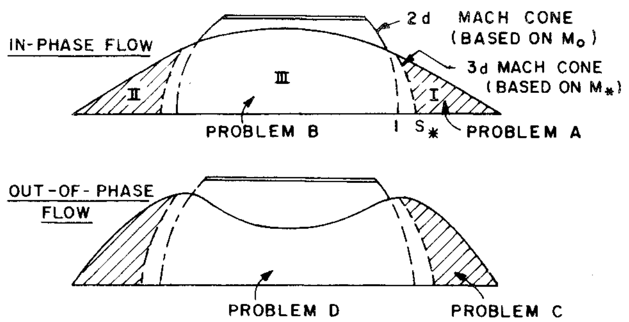

Fig. 1b Problems A, B, C, and D and the defining regions.

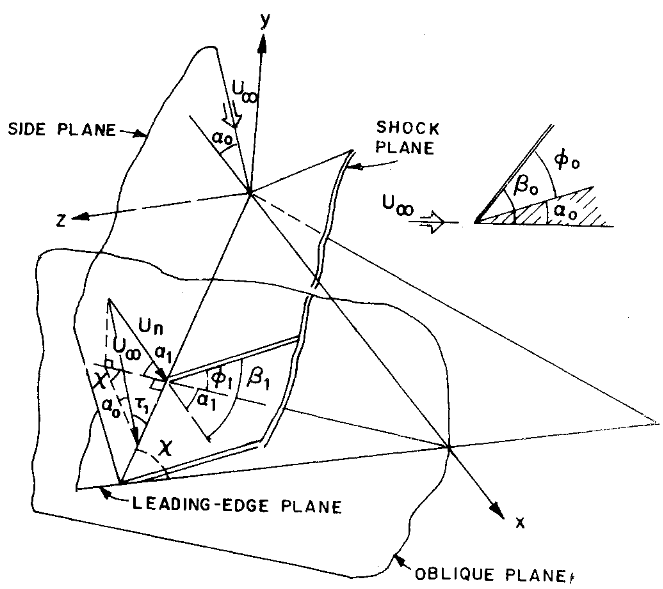

Fig. 1c Delta wing showing notations.

which require the conservation of mass (3a), momentum (3b)-(3d) and energy (3e). The bracket represents the shock jump difference. It is defined as $\langle \cdot \rangle = (\cdot)_\infty - (\cdot)_a$, where $(\cdot)_\infty$ denotes the uniform-flow at the upstream states of the shock wave and $(\cdot)_a$ denotes the downstream states immediately behind the shock wave. The vectors τ and s represent two orthogonal tangential directions to the shock surface whose equation is $\bar{G}(\hat{r},t)=0$. The formulated problem therefore is a free boundary problem in that the shock surface $\bar{G}=0$, unknown in advance, is to be determined as a part of the solution.

Enclosed between the shock surface and the wing surface, the flow region is divided in two parts by the three dimensional Mach cone as given by Eq. (6). The outer region (shaded area) is defined from the cone extended outboard toward the leading edges, whereas the inner region is enclosed by the cone (Fig. 1a). It should be noted that the 3-d Mach cone is defined by the skewed-wedge Mach number, M_* [see Eq. (69)] and the 2-d Mach cone is defined by the two-dimensional wedge Mach number M_0 (Fig. 1b); hence M_* becomes M_0 as χ approaches zero. In the outer region, the steady flowfield is a uniform one. Hence, the flow velocities and properties have exact solutions, i.e.

$$u_* = u_0\,(1+u_s) \qquad p_* = p_0\left(1+\frac{\gamma M_0^2}{\lambda}p_s\right)$$
$$v_* = u_0 v_s$$
$$w_* = u_0\, w_s \qquad \rho_* = \rho_0\left(1+\frac{M_0^2}{\lambda}\rho_s\right) \tag{4a}$$

where

$$u_s = \frac{\cos\phi_0}{\cos\beta_0}\cdot\left[\cos\tau_1\,\sin\chi + \frac{\sin\tau_1\,\cos\beta_1}{\cos\phi_1}\right] - 1$$
$$v_s = 0$$
$$w_s = \frac{\cos\phi_0}{\cos\beta_0}\left[\cos\tau_1\cos\chi - \frac{\sin\tau_1\,\cos\beta_1}{\cos\phi_1}\sin\chi\right]$$
$$p_s = \frac{2\lambda M_\infty^2}{M_0^2}\left[\frac{\sin^2\beta_1\sin^2\tau_1 - \sin^2\beta_0}{2\gamma M_\infty^2\sin^2\beta_0 - (\gamma-1)}\right]$$
$$\rho_s = \frac{2\lambda}{M_0^2\sin^2\beta_0}\left[\frac{\sin^2\beta_1\sin^2\tau_1 - \sin^2\beta_0}{2+(\gamma-1)M_\infty^2\sin^2\tau_1\sin^2\beta_1}\right]$$

In Eqs. (4a) and (4b), all expressions with subscript $(\,)_0$ represent the aft-shock flow properties and velocities corresponding to the case of a plane wedge flow with semi-wedge angle equal to α_0. In the side plane (Fig. 1c), the angle τ_1, can be defined as $\cos\tau_1 = \cos\alpha_0\,\sin\chi$. The flow deflection angle α_i and the wave angle β_i denote the plane wedge flow when $i=0$ and the skewed wedge flow when $i=1$ (contained in the oblique plane), whereas $\phi_i = \beta_i - \alpha_i$. The well-known oblique shock relation $\beta_i = \beta_i(\alpha_i)$ can be found elsewhere (e.g. Ref. 4).

Furthermore, the steady oblique plane shock surface is described by

$$\bar{G}_* \equiv \bar{y} - \tan\phi_1\,(\bar{x}\,\cos\chi - \bar{z}\sin\chi) = 0 \tag{5}$$

and the three dimensional Mach cone is described by

$$(M_*^2 - 1)\,[\bar{y}^2 + (\bar{x}\,\sin\theta_* - \bar{z}\,\cos\theta_*)^2] = (\bar{x}\,\cos\theta_* + \bar{z}\,\sin\theta_*)^2 \tag{6a}$$

where

$$\theta_* = \tan^{-1}\left(\frac{w_*}{1+u_*}\right) \tag{6b}$$

is the 3d cone angle measured between the x-axis and the outer flow direction in regions I and II (Fig. 1b), and M_* is the aft-shock Mach number due to the skewed wedge flow.

Bounded by the Mach cone and the curved shock wave, the inner region, on the other hand, is one which contains nonuniform flow. Clearly, for the case of steady flow at incidence α_0 (referred to as mean flow hereafter), the flow in both regions is conical. However since there is one more independent variable involved in the unsteady formulation, conical flow can no longer be assumed. Also, it should be remarked that only the outer mean flow is irrotational; the flowfield, steady or unsteady, is otherwise rotational.

III. Problem Formulation

Perturbation Schemes

In the unsteady treatment, we consider that the wing performs oscillatory pitching motion around the mean incidence α_0 with an amplitude $\bar{\epsilon}$ and at a certain circular frequency $\bar{\omega}$. The normalized amplitude ϵ and the reduced frequency k are assumed much smaller than unity and are defined as follows

$$\epsilon = \bar{\epsilon}/\alpha_0 << 1$$

and

$$k = \bar{\omega}L/U_\infty << 1$$

where L is the root chord of the wing. Two other reduced frequencies of the same order as k can be defined i.e.,

$$k_* = \bar{\omega}L/U_* \quad \text{and} \quad k_0 = \bar{\omega}L/u_0$$

Obviously, k_* approaches k_0 as u_* approaches u_0 and both are much smaller than unity since $u_0,\ u_* \sim 0(U_\infty)$. Our perturbation scheme consists of two stages:

First, based on the linear perturbation scheme of small amplitude, the solution is sought in the following form, i.e.,

$$\left.\begin{aligned}
\bar{p}(\hat{r},\bar{t}) &= p_\Delta(\hat{r}) + \epsilon\cdot p_\Delta P(\hat{r},\bar{t};\delta) + \ldots\\
\bar{u}(\hat{r},\bar{t}) &= u_\Delta(\hat{r}) + \epsilon\cdot u_\Delta U(\hat{r},\bar{t};\delta) + \ldots\\
\bar{v}(\hat{r},\bar{t}) &= \qquad\quad \epsilon\cdot u_\Delta V(\hat{r},\bar{t};\delta) + \ldots\\
\bar{w}(\hat{r},\bar{t}) &= w_\Delta(\hat{r}) + \epsilon\cdot u_\Delta W(\hat{r},\bar{t};\delta) + \ldots\\
\bar{\rho}(\hat{r},\bar{t}) &= \rho_\Delta(\hat{r}) + \epsilon\cdot \rho_\Delta R(\hat{r},\bar{t};\delta) + \ldots
\end{aligned}\right\} \tag{7a}$$

$$\bar{G}(\hat{r},\bar{t}) = \bar{G}_\Delta(\hat{r}) + \epsilon\bar{Q}(\hat{r},\bar{t}) \tag{7b}$$

where subscript $(\cdot)_\Delta$ denotes the mean flow solutions previously obtained by Hui.[10] For example, in the outer region, $p_\Delta(\hat{r})$ and $\bar{G}_\Delta(\hat{r})$ become the exact solution of p_* and $\bar{G}_*$.

When the amplitude perturbation scheme, Eq. (7), is introduced into the Eqs. (1, 2, and 3), and terms $0(\epsilon^2)$ neglected, the following set of equations and boundary conditions can be obtained.

The differential equations (D.E.) are:

$$\Box_*^2 P + M_*^2 a[2(P_{xz}+P_{zt}) + aP_{zz}] = 0 \tag{8a}$$

$$P_t + P_x + aP_z = -\gamma(U_x + V_y + W_z) \tag{8b}$$

$$U_t + U_x + aU_z = -\nu P_x \tag{8c}$$

$$V_t + V_x + aV_z = -\nu P_y \tag{8d}$$

$$W_t + W_x + aW_z = -\nu P_z \tag{8e}$$

$$\gamma(R_t + R_x + aR_z) = P_t + P_x + aP_z \tag{8f}$$

where Eq. (8a) results from Eqs. (8b-8e), and $\Box_*^2$ is the "acoustic" operator defined as

$$\Box_*^2 = (M_*^2 - 1)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + 2M_*^2\frac{\partial^2}{\partial x\partial t} + M_*^2\frac{\partial^2}{\partial t^2}$$

$$a = w_*/u_* \qquad \nu = 1/\gamma M_*^2$$

and, the coordinate systems are normalized according to $(x,y,z,t) = (\bar{x}/L,\ \bar{y}/L,\ \bar{z}/L,\ \bar{t}\bar{u}_*/L)$ (hence $r = \hat{r}/L$).

The flow tangency condition (T.C.), evaluated on the wing surface $\bar{S}=0$, reads

$$V=\frac{\partial\Delta}{\partial x}+\frac{\partial\Delta}{\partial t} \quad \text{at} \quad y=0 \tag{9}$$

The shock condition (S.C.), evaluated on the mean shock surface $\bar{G}_\Delta=0$, reads

$$\begin{Bmatrix} V \\ P \\ U \\ W \\ R \end{Bmatrix} = [\mathbf{S}]\cdot \begin{Bmatrix} Q_x \\ Q_z \\ Q_t \end{Bmatrix} \tag{10a}$$

at $y=(x\cos\chi - z\sin\chi)\tan\phi_I$

where the coefficient matrix [S] is obtained after much algebraic manipulation from Eqs. (3a-3e).

$$[\mathbf{S}] = \begin{bmatrix} \tilde{A}_I & \tilde{A}_2 & \tilde{B} \\ \tilde{C}_I & \tilde{C}_2 & \tilde{D} \\ \tilde{E}_I & \tilde{E}_2 & \tilde{F} \\ \tilde{K}_I & \tilde{K}_2 & \tilde{L} \\ \tilde{G}_I & \tilde{G}_2 & \tilde{J} \end{bmatrix} \tag{10b}$$

Here, we omit the expressions of the coefficients in [S], as they are quite involved and can be found in Ref. 20. It should be noted that [S] reduces to Hui's wedge coefficient matrix when $\chi\to 0$, as expected.[12]

The oscillatory motion of the wing is restricted by the shock attachment condition (S.A.C.), i.e.,

$$Q(x,y,t)=x-x_0 \quad \text{at} \quad z=x\cot\chi \tag{11}$$

where $Q=\bar{Q}/L$.

Secondly, based on a linear perturbation scheme of small reduced-frequency k_*, P, U... etc. can be expanded in series in ik_* and split into the in-phase and the out-of-phase components, i.e.,

$$P(r,t)=[P^0(r)+ik_*P^I(e)+0(k_*^2)]e^{ik_*t} \tag{12a}$$

$$Q(r,t)=[Q^0(r)+ik_*Q^I(r)+0(k_*^2)]e^{ik_*t} \tag{12b}$$

Similar expressions follow for the other variables U, V, W, and R. The general formulation [Eq. (8-12)] is equally applicable for the outer and inner regions. If the shock attachment condition Eq. (11) is replaced by the Mach cone condition, Eq. (14d), then the inner-region formulation is contained in the formulation as a special case when "a" is set to zero. Since a is equal to w_*/u_* and w_* is proportional to $\tan\chi$. As "a" vanishes, all the basic, nonperturbed flow velocities and properties reduce from those of the skewed wedge flow to those of the two-dimensional wedge flow [e.g. all ( )s terms vanish in Eq. (4a)]. Hence, in the inner formulation the effect of sweepback angle χ only comes in from the Mach cone condition. Thus, in the inner region, $P(r,t)$ and $Q(r,t)$ reduce to $\hat{P}(r,t)$ and $\hat{Q}(r,t)$, where

$$\hat{P}(r,t)=[P^{(0)}(r)+ik_0P^{(I)}(r)+0(k_0^2)]e^{ik_0t}$$

$$\hat{Q}(r,t)=[Q^{(0)}(r)+ik_0Q^{(I)}(r)+0(k_0^2)]e^{ik_0t}$$

Consequently, the in-phase component becomes $P^{(0)}(r)$ and $Q^{(0)}(r)$, and the out-of-phase component becomes $P^{(I)}(r)$ and $Q^{(I)}(r)$. [Actually, some factors are involved, for example, in converting the outer P^0 into the inner $P^{(0)}$ — see Eqs. (14)].

Problems A,B,C, and D

For the convenience of later referral, we divide the problem into four parts (Fig. 1b). Problems A and B, that of calculating P^0 and $P^{(0)}$, are the in-phase problems; both are conical flow problems. Problems C and D, that of calculating P^I and $P^{(I)}$, are the out-of-phase problems; neither is a conical flow problem. In the crossflow plane, problems A and C are governed by equations of hyperbolic type whereas problems B and D, that of elliptic type.

It is noted that within the framework of the linear amplitude perturbation the outer flow problem is based on an exact formulation. Meanwhile, the inner formulation is an approximate one in that terms of $0(\epsilon\delta)$ are neglected consistently in the equations as well as the boundary conditions. In essence, the approximation ignores the interaction between the mean flow and the out-of-phase contribution of the reference wedge flow (Ref. 20, Appendix H). However, the terms ignored can be recovered, if one wishes, in a straightforward manner following a successive approximation scheme[20] taking the present formulation as its first order approximation. In fact, the terms ignored will eventually turn out to be the known inhomogeneous terms in the equations and the boundary conditions of next higher order. Thus, the present approximation should expect to give accurate results at least for cases in which: 1) the mean flow incidence and/or the sweepback angle is not too large where the effect of interaction is small; 2) the Mach number is sufficiently high so that the outer flow dominates the inner one over the planform.

Let us now introduce the frequency perturbation, Eqs. (12), to the time-dependent formulation, Eqs. (8-11). For problems A and C, we obtain the following formulation: problems A and C

D.E.

$$\lambda_*^2P_{xx}^n - P_{zz}^n + aM_*^2(2P_{xz}^n + aP_{zz}^n) = n\cdot[-2M_*^2(P_x^{n-I}+aP_z^{n-I})]$$

$$\left.\begin{aligned} &P_x^n + aP_z^n + \gamma(U_x^n + V_y^n + W_z^n) = n\cdot[-P^{n-I}] \\ &U_x^n + aU_z^n + \nu P_x^n = n\cdot[-U^{n-I}] \\ &V_x^n + aV_z^n + \nu P_y^n = n\cdot[-V^{n-I}] \\ &W_x^n + aW_z^n + \nu P_z^n = n\cdot[-W^{n-I}] \end{aligned}\right\} \tag{13a}$$

T.C.

$$V^n = n(x-x_0)-(n-I) \quad \text{at} \quad y=0 \tag{13b}$$

S.C.

$$\left.\begin{aligned} &P^n = \tilde{C}_IQ_x^n + \tilde{C}_2Q_z^n + n\cdot\tilde{D}Q^{n-I} \\ &U^n = \tilde{E}_IQ_x^n + \tilde{E}_2Q_z^n + n\cdot\tilde{F}Q^{n-I} \\ &V^n = \tilde{A}_IQ_x^n + \tilde{A}_2Q_z^n + n\cdot\tilde{B}Q^{n-I} \\ &W^n = \tilde{K}_IQ_x^n + \tilde{K}_2Q_z^n + n\cdot\tilde{L}Q^{n-I} \end{aligned}\right\} \tag{13c}$$

at $y=\tan\phi_I\,(x\cos\chi - z\sin\chi)$

S.A.C.

$$Q^n = n\cdot(x-x_0) \quad \text{at} \quad z=x\cot\chi \tag{13d}$$

Case $n=0$ corresponds to problem A; case $n=I$ corresponds to problem C.

On the other hand, if we now introduce the following transformations to the Eqs. (13)

$$(\xi,\eta,\zeta)=(x,\lambda y/x,\lambda z/x) \tag{14a}$$

$$(P^{(n)}, V^{(n)}, W^{(n)}, Q^{(n)}) = (P^n/\tilde{\alpha}, V^n/\tilde{\beta}, W^n/\tilde{\beta}, Q^n/\lambda) \quad (14b)$$

problems A and C can be recast into the inner flow problems of B and D. Thus, letting $a=0$, we obtain: problems B and D

D.E.

$$n\cdot[\xi^2 P_{\xi\xi}^{(n)} - 2\xi(\eta P_{\eta\xi}^{(n)} + \zeta P_{\zeta\xi}^{(n)})] + \nabla_2 P^{(n)} + 2\nabla_1 P^{(n)} = n\cdot[-2\kappa^2\xi\nabla_1 P] \quad (14c)$$

$$\left.\begin{aligned} &n\cdot\xi P_\xi^{(n)} - \nabla_1 P^{(n)} + V_\eta^{(n)} + W_\zeta^{(n)} = n\cdot\xi[U^{(n-1)}/\lambda - \kappa^2 P^{(n-1)}] \\ &n\cdot\xi V_\xi^{(n)} - \nabla_1 V^{(n)} + P_\eta^{(n)} = n\cdot\xi[-V^{(n-1)}] \\ &n\cdot\xi W_\xi^{(n)} - \nabla_1 W^{(n)} + P_\zeta^{(n)} = n\cdot\xi[-W^{(n-1)}] \end{aligned}\right\}$$

T.C.

$$P_\eta^{(n)} = n\cdot[-2\xi] \quad \text{at} \quad \eta=0 \quad (14d)$$

S.C.

$$\left.\begin{aligned} &P^{(n)} = C[G^{(n)} - \zeta G_\zeta^{(n)} + n\cdot\xi G_\xi^{(n)}] + D\cdot[n\cdot\xi G^{(n-1)}(\zeta)] \\ &V^{(n)} = A[G^{(n)} - \zeta G_\zeta^{(n)} + n\cdot\xi G_\xi^{(n)}] + B\cdot[n\cdot\xi G^{(n-1)}(\zeta)] \\ &W^{(n)} = KG_\zeta^{(n)} \quad \text{at} \quad \eta=H \end{aligned}\right\} \quad (14e)$$

where the symbols ∇_1 and ∇_2 are differential operators defined as

$$\nabla_1 = \eta\frac{\partial}{\partial\eta} + \zeta\frac{\partial}{\partial\zeta}$$

$$\nabla_2 = (\eta^2-1)\frac{\partial^2}{\partial\eta^2} + 2\eta\zeta\frac{\partial^2}{\partial\eta\partial\zeta} + (\zeta^2-1)\frac{\partial^2}{\partial\zeta^2}$$

Clearly, the coefficients of Eq. (14c) are related to those of Eq. (13c) as $(A,B,C,D,K) = (\tilde{A}_1/\lambda, \tilde{B}_1/\lambda, \nu_0\tilde{C}_1, \nu_0\tilde{D}_1, \tilde{K}_2)$. This time, case $n=0$ corresponds to problem B; case $n=1$ corresponds to problem D. The transformed shock shape is defined as $G^{(n)} = Q^{(n)}/x$, which is a function of ξ and ζ for the case of $n=1$ and of ζ alone for the case of $n=0$. Lastly, to complete the boundary value problem, the Mach cone condition (M.C.) is introduced. The M.C. simply requires that all the dependent variables (but not their derivatives) must be matched to those from the outer region, i.e.,

M.C.

$$(P_*^{(n)}, V_*^{(n)}, W_*^{(n)}) = (P_*^n/\tilde{\alpha}, V_*^n/\tilde{\beta}, W_*^n/\tilde{\beta}) \quad \text{at} \quad \eta^2+\zeta^2=1 \quad (14f)$$

where $(\ )_*$ denotes that the variables are evaluated at the cone surface.

IV. Methods of Solution

Other than obtaining solutions from the previous formulation of problems A and B, an alternative method may be employed to obtain the in-phase solutions. Since Hui's mean flow solution, say p_Δ, generally contains α_0 as a parameter, the in-phase solution is simply its derivative $\partial p_\Delta/\partial\alpha_0$. This solution is equivalent to solving problems A and B and patching solution B to solution A (Ref. 20, p. 37) in the same manner as Hui's mean flow treatment. For this reason, only a brief account on the in-phase part will be given; rather, more efforts will be devoted to the solution methods for the out-of-phase problems.

Exact Solutions: Problems A and C

Careful inspection of Eqs. (13) reveals that exact solutions can be found in the simplest possible forms:

In problem A, $n=0$, we have

$$\left.\begin{aligned} P^0 &= \tilde{C}_1 d_5 + \tilde{C}_2 e_5 \\ V^0 &= 1 \\ U^0 &= \tilde{E}_1 d_5 + \tilde{E}_2 e_5 \\ W^0 &= \tilde{K}_1 d_5 + \tilde{K}_2 e_5 \end{aligned}\right\} \quad (15a)$$

and the shock shape is an oblique planar surface described by

$$Q^0(x,z) = d_5 x + e_5 z - x_0 \quad (15b)$$

where

$$d_5 = \frac{1-\tilde{A}_2 \tan\chi}{\tilde{A}_1 - \tilde{A}_2 \tan\chi}$$

and

$$e_5 = \tan\chi(1-d_5)$$

For problem C, $n=1$, solutions of the following form are sought, i.e.,

$$\begin{bmatrix} P^1 \\ V^1 \\ U^1 \\ W^1 \end{bmatrix} = [\mathbf{C}] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \text{ where } [\mathbf{C}] = \begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{bmatrix} \quad (16a)$$

$$Q^1(x,z) = A_5 x^2 + B_5 xz + C_5 z^2 + D_5 x + E_5 z + F_5 \quad (16b)$$

Substituting Eqs. (16a) and (16b) into Eqs. (13a-13d) yields uniquely twenty-two coefficients A_1, B_1..., F_5. They are provided by twenty-two algebraic equations generated by Eqs. (13a and 13b). Since the solution procedure is straightforward, but rather laborious, we shall not list them explicitly here.‡ Detailed expressions of these coefficients are referred to in Ref. 20 (p. 76). A special wedge case (Hui[12]) is also contained in Eq. (16), when $\chi\to 0$, as expected.

From Eq. (16a), it can be seen that the out-of-phase pressure, density, and flow velocities are found to be linear functions of (x,y,z). In fact, they assume constant values on the prescribed parallel planar surfaces (called isobar and isovelocity surfaces). Similar to the nature of the conical rays for the steady flow, a quasiconical surface is found to exist for the out-of-phase flow. Thus, the "quasiconical" flow is defined in the following manner. That is, suppose the wing oscillates around the wing apex, P^0 and P^1 can be generally recast to a form as

$$P^n = x^n \Phi^n(y/x,\ z/x)$$

While P^0 is conical when $n=0$, P^1 is quasiconical when $n=1$. This observation then initiates a related solution structure for the problem D. In passing, it is worthy noting that the quasiconical potential flowfield has been defined previously by Lomax and Heaslet[22] in the steady wing theory and was mentioned even earlier (but without proof) for stability calculation by Malvestuto and Margolis.[23] Nevertheless, these cases hold only for the linearized supersonic flow potential. As a generalization, the present analysis has shown that the quasiconical flowfield indeed exists for the pressure formulation, which also includes the shock boundary conditions.

Quasiconical Formulation

We intend to leave out the description of problem B (details are given in Ref. 20), as it is nearly identical to Hui's method of solution for the mean flow.[10] Moreover, the method used to solve problem D subsequently also contains a solution method for problem B. In what follows, therefore, we suppose the solution method for problem (B) such as $P^{(0)}$, $V^{(0)}$ and $W^{(0)}$ are known functions of η and ζ and the shock shape $G^{(0)}$ is a known function of ζ.

‡We note that the coefficients D_1, D_2, D_3, D_4 and F_5 are proportional to x_0.

Motivated by our earlier observation, we seek the solution (D) in the following form, i.e.,

$$\left.\begin{aligned} P^{(1)} &= \xi\Phi_1(\eta,\zeta) + x_0[\Phi_0(\eta,\zeta) - \lambda D] \\ V^{(1)} &= \xi\Psi_1(\eta,\zeta) + x_0[\Psi_0(\eta,\zeta) - \lambda B] \\ W^{(1)} &= \xi\Theta_1(\eta,\zeta) + x_0\Theta_0(\eta,\zeta) \end{aligned}\right\} \tag{17a}$$

and the shock shape is sought in the form of

$$G^{(1)} = \xi T_1(\zeta) + x_0 T_0(\zeta) \tag{17b}$$

In the absence of the second terms, and for convenience, Φ_1, Ψ_1... will be called the "apex" problem, whereas the second terms Φ_0, Ψ_0,... called the "pivot" problem, indicating that they add the varying pivot effect to the total solution. Hence substituting Eqs. (17) into Eqs. (14) again yields a compact representation for these two sets of problems; namely, the case of $m=1$ corresponds to the apex problem and the case of $m=0$ corresponds to the pivot problem, i.e.,

$$\text{D.E.} \quad [\nabla_2 + 2(1-m)\nabla_1]\Phi_m = m\cdot[-2\kappa^2\nabla_1 P^{(0)}] \tag{18a}$$

$$\text{T.C.} \quad \frac{\partial\Phi_m}{\partial\eta} = m\cdot[-2] \qquad \text{at} \quad \eta=0 \tag{18b}$$

$$\begin{aligned} \text{S.C.} \quad \Phi_m &= C[(1+m)T_m(\zeta) - \zeta T_m'(\zeta)] + m\cdot DG^{(0)}(\zeta) \\ \Psi_m &= A[(1+m)T_m(\zeta) - \zeta T_m'(\zeta)] + m\cdot BG^{(0)}(\zeta) \\ \Theta_m &= KT_m'(\zeta) \qquad \text{at} \quad \eta = H \end{aligned} \tag{18c}$$

$$\text{M.C.} \quad \begin{pmatrix} \Phi_m \\ \Psi_m \\ \Theta_m \end{pmatrix} = m\cdot[\mathbf{E}]\,[\mathbf{e}] + (1-m)\cdot[\mathbf{d}] \qquad \text{at} \quad \eta^2+\zeta^2=1 \tag{18d}$$

where

$$[\mathbf{E}] \rightarrow [\mathbf{E}_*] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \equiv \begin{bmatrix} \bar{\alpha}A_1 & \bar{\alpha}B_1/\lambda & \bar{\alpha}C_1/\lambda \\ \bar{\beta}A_2 & \bar{\beta}B_2/\lambda & \bar{\beta}C_2/\lambda \\ \bar{\beta}A_3 & \bar{\beta}B_3/\lambda & \bar{\beta}C_3/\lambda \end{bmatrix} \tag{18e}$$

$$[\mathbf{e}] \rightarrow [\mathbf{e}_*] = \begin{pmatrix} 1 \\ \eta^* \\ \zeta^* \end{pmatrix} \tag{18f}$$

$$[\mathbf{d}] \rightarrow [\mathbf{d}_*] = \frac{1}{x_0}\begin{pmatrix} \bar{\alpha}d_1 + \lambda D x_0 \\ \bar{\beta}d_2 + \lambda B x_0 \\ \bar{\beta}d_3 \end{pmatrix} \tag{18g}$$

The Mach cone condition Eqs. (18d-18g) requires some explanations. We notice that Eqs. (18) is formulated primarily as a boundary value problem, whose solutions are to be evaluated at the wing surface and at two other approximate boundaries, namely the two-dimensional shock height and the two-dimensional Mach cone, based on a "linear" approximation up to $0(\epsilon\delta)$. Since the outer problem possesses exact solutions and the exact Mach cone surface is known [Eq. (6)], it is tempting to patch the inner solution with the outer solution at the exact boundaries. Thus, the "linear" matrices [**E**], [**e**] and [**d**] need not be known; instead, they are to be replaced by the exact matrices [$\mathbf{E}_*$], [$\mathbf{e}_*$] and [$\mathbf{d}_*$] given in Eqs. (18e-18g) respectively. It is noticed that in [$\mathbf{e}_*$], η^* and ζ^* are determined by the exact Mach cone given by Eq. (6). Similar patching will be applied to the S.C. in the next section. Lastly, the flow symmetry demands $\partial\Phi_m/\partial\zeta=0$ at $\zeta=0$.

It is seen that the problem is reduced completely to the conical problem and it can be further formulated in terms of one dependent variable, Φ_m, alone by eliminating the unknown shock function T_m. Nevertheless, the shock condition for Φ_1 in this case involves second-order derivatives. Some care, therefore, is required in obtaining the solution of such a problem. On the other hand, since our aim is to evaluate global quantities such as the pitching moment, the pressure solutions such as $\Phi_1(\eta,\zeta)$ need not be solved if the other direct global method becomes applicable. Indeed, a spanwise integral method, worked out by Malmuth[9] and Hui[11] for area-rule consideration, is found most suitable for this purpose. Thus, when the method is applied to Eqs. (18), an ordinary differential equation results with two end point conditions, one being at the shock, the other at the wing surface.

Integral Solutions: Problems B and D

The spanwise integration technique has been used by Miles[24] and Frochlich[25] for the purpose of reduction of one variable in their unsteady potential flow formulation. Such a technique has been adopted by Malmuth[9] and Hui[11] for direct evaluations of steady forces and moments with attached shock flow. Again, such an approach can be generalized for the treatments of our problems B and D.

Instead of the half range integration used previously,[9,11] the full range spanwise integral is introduced to cover the total range of the Mach cone. Define the operator I as follows.

$$I[\Phi_1(\eta,\zeta)] = \int_{-\sqrt{1-\eta^2}}^{\sqrt{1-\eta^2}} \Phi_1(\eta,\zeta)\,d\zeta = F(\eta) \tag{19}$$

Similar integrals can be defined for $\Psi_1(\eta,\zeta)$ and $\Theta_1(\eta,\zeta)$. Since the integrals are in full range, only those involving even integrands are to be evaluated, whereas those involving odd integrands are identical zero due to antisymmetry. Without loss of generality, we now apply the integral transformation Eq. (19) to Eqs. (18). By making use of some properties of the integral on the Mach cone (Appendix E, Ref. 20) and after some arrangement, we obtain the following ordinary differential equation formulation.

$$\begin{aligned} \text{D.E.} \quad & (1-\eta^2)F''(\eta) + 2\eta F'(\eta) - 2F(\eta) \\ &= \alpha_I \frac{\eta}{\sqrt{1-\eta^2}} + \alpha_{II}\frac{1}{\sqrt{1-\eta^2}} + \alpha_{III} \end{aligned} \tag{20a}$$

$$\text{T.C.} \qquad F'(0) = -4 \tag{20b}$$

$$\text{S.C.} \qquad F'(H) + 2\left(\frac{A_0+H}{1-H^2}\right)F(H) = \sigma_I + \sigma_{II} \tag{20c}$$

where

$$\alpha_I = -6b_1,\ \alpha_{II} = -6a_1,\ \alpha_{III} = -6c_1 - 4\kappa^2\left(\frac{HW_*^{(0)}}{A_0+H}\right)$$

$$\sigma_I = \frac{2}{H'}(\Psi_1^* - H\Theta_1^*)$$

$$\sigma_{II} = \frac{2}{H'}\left\{\left[H\frac{E_0}{\lambda} - \kappa^2\frac{2}{C}(B-A_0D) - A_0\right] \times \left[P_*^{(0)} - \frac{HW_*^{(0)}}{H'(A_0+H)}\right] - 2H'(B-A_0D)G_*^{(0)}\right\}$$

$$A_0 = A/C \text{ and } E_0 = E/C \tag{20d}$$

and the terms with subscript $(\ )_*$ and with superscript $(\ )^*$ are again terms that are patched with solutions A, i.e,

$$\left.\begin{aligned} &P_*^{(0)},\ W_*^{(0)} = P^0,\ W^0 \text{ of Eq. (15a)} \\ &G_*^{(0)} \equiv G^{(0)}(\zeta^*) \text{ is to replace } G^{(0)}(H') \end{aligned}\right\} \qquad (20e)$$

and with solution C, i.e.,

$$\left.\begin{aligned} \Phi_I^* &= a_I + b_I\eta^* + c_I\zeta^* \\ \Theta_I^* &= a_3 + b_3\eta^* + c_3\zeta^* \end{aligned}\right\} \qquad (20f)$$

Thus, a closed form solution of Eq. (20) can be found, i.e.,

$$F(\eta) = c_I\eta + c_{II}(1+\eta^2) - \frac{\alpha_I}{3}\sqrt{1-\eta^2} - \frac{\alpha_{II}}{3}\eta\sqrt{1-\eta^2} - \frac{\alpha_{III}}{2} \qquad (21)$$

where

$$c_I = \frac{\alpha_{II}}{3} - 4$$

$$c_{II} = \frac{H'^2}{2(HH'^2 + (A_0+H)(1+H^2))} \cdot \Big\{\sigma_I + \sigma_{II} - \frac{1}{H'}\Big[c_I\Big(H' + 2\frac{H(A_0+H)}{H'}\Big) + \frac{\alpha_I}{3}(2A_0+H) + \frac{\alpha_{II}}{3} \times (2A_0H+1) + \alpha_{III}\frac{A_0+H}{H'}\Big]\Big\}$$

The pivot problem can be solved essentially in the same way. Spanwise integral for $\Phi_o(\eta,\zeta)$ can be defined as

$$I[\Phi_0(\eta,\zeta)] = f(\eta)$$

The ordinary differential equation formulation then reads,

$$\text{D.E.}\quad f''(\eta) = -2\Phi_0^*/(1-\eta^2)^{3/2} \qquad (22a)$$

$$\text{T.C.}\quad f'(0) = 0 \qquad (22b)$$

$$\text{S.C.}\quad f'(H) + \frac{A_0+H}{1-H^2}f(H) = \frac{2A_0}{H'}\Phi_0^* + \frac{2H}{H'^2}\Theta_0^* \qquad (22c)$$

where $\Phi_0^* = C\dfrac{\lambda D_5}{x_0}$, $\quad \Theta_0^* = K\dfrac{\lambda E_5}{x_0}$

The solution of Eq. (22a) reads

$$f(\eta) = 2\Big[\Phi_0^*\sqrt{1-\eta^2} - \frac{H\Theta_0^*}{A_0+H}\Big] \qquad (23)$$

Hence, the total spanwise-integral pressure for the problem D reads

$$I[P^{(1)}] = \xi F(\eta) + x_0(f(\eta) - 2\lambda D\sqrt{1-\eta^2}) \qquad (24)$$

We note that Eqs. (20) and (22) are basically "linear" formulations based on the $0(\epsilon)$ approximation. The patching procedure used here is to replace all the linear values of these inhomogeneous terms by their exact quantities at the Mach cone [denoted by $(\ )_*$ or $(\ )^*$]. This is done only to the extent that the dependent variables are subject to the patching but the inner shock coordinate such as H (and H') remain unaltered. In this sense, the patching procedure used here is a hybrid approximation, which is not pertinent to the method of strained coordinates.

Finally, we believe that the local pressure field if needed can be generated with the help of present integral results. A part of the solution method should be similar to the integral series representation developed by Malmuth[21] for the mean flow case [Eqs. (4a) and (4b)], in a sense that the coefficient of the leading term of the assumed pressure series can be readily evaluated from the known spanwise integral. The proposed formulation for solving the local pressure field was outlined in Ref. 20 (page 88-90).

V. Stability Derivatives

The in-phase $(j=0)$ and the out-of-phase $(j=1)$ moment derivatives (called the stiffness and the damping moments) are

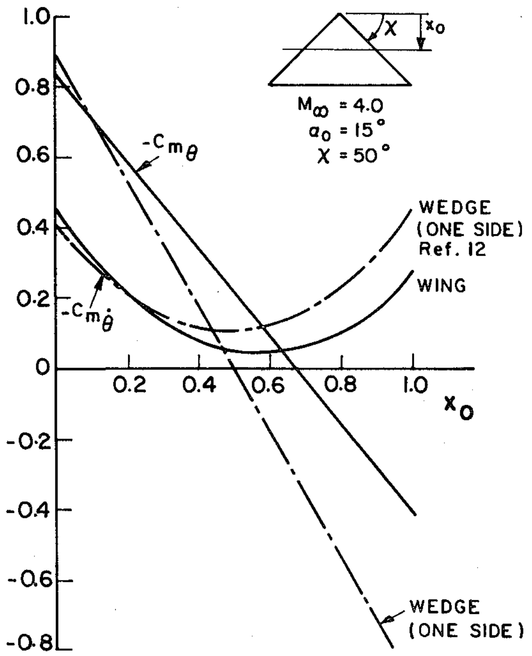

Fig. 2 Variation of $-C_{M_\theta}$ and $-C_{M_{\dot\theta}}$ with pitching axis location x_0.

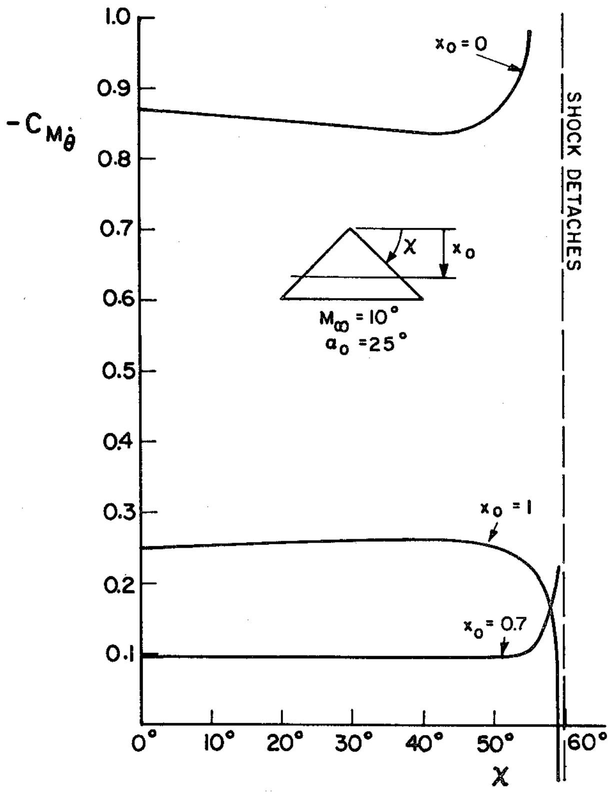

Fig. 3 Variation of $-C_{M_{\dot\theta}}$ with sweepback angle χ.

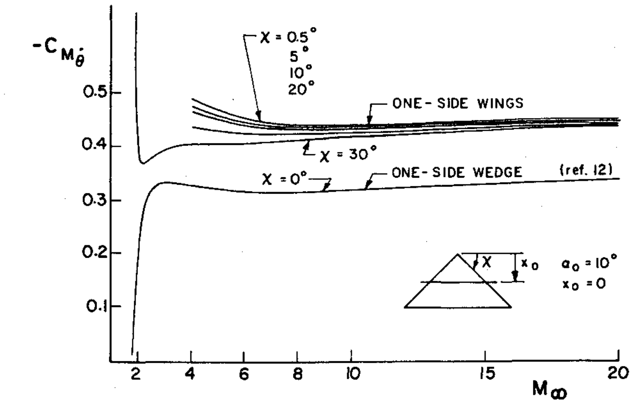

Fig. 4 Variation of $-C_{M_{\dot\theta}}$ with freestream Mach number M_∞.

defined as

$$C_{M_\mu}=\frac{2}{\Lambda}\int_{x=0}^{x=1}\int_{z=0}^{z=\Lambda x}(x-x_0)(C_{p_\mu}{}^{(j)}+\Phi_{,_\mu}{}^{(j)})\mathrm{d}z\mathrm{d}x \tag{26}$$

where $\mu=(1-j)\theta+j\dot\theta$

$$C_{p_\mu}{}^{(j)}=2\left(\frac{\nu_\infty\bar p_0}{\nu_0\lambda}\right)[(1-j)+j\bar u_0]\cdot p^{(j)} \qquad 0\le z\le x\Lambda_*$$
$$=0 \qquad \text{otherwise} \tag{26a}$$

$$C_{p_\mu}{}^{j}=2(\nu_\infty\bar p_*)[(1-j)+j\bar u_*]\cdot P^j \qquad x\Lambda_*\le z\le x\Lambda$$
$$=0 \qquad \text{otherwise}$$

Hence, the damping moment can be integrated to yield the following algebraic expressions, i.e.,

$$C_{M_{\dot\theta}}=C_{M_{\dot\theta}}{}^{(l)}+C_{M_{\dot\theta}}{}^{l} \tag{27a}$$

where

$$-C_{M_{\dot\theta}}{}^{(l)}=\left[\frac{2\zeta^*}{\Lambda\lambda^2}\cdot\left(\frac{\nu_\infty}{\nu_0}\right)(\bar p_0\bar u_0)\right]\left\{\frac{1}{4}F(0)+\frac{x_0}{3}[(f(0)-F(0)]-\frac{x_0^2}{2}f(0)\right\} \tag{27b}$$

$$-C_{M_{\dot\theta}}{}^{l}=\left[\frac{2}{3}\left(1-\frac{\Lambda_*}{\Lambda}\right)\nu_\infty\bar p_*\bar u_*\right]\cdot\left\{\frac{3}{2}\left[A_l+\frac{C_l}{2}(\Lambda+\Lambda_*)\right]-x_0[2(A_l-d_0)+C_l(\Lambda+\Lambda_*)]-3x_0^2\cdot d_0\right\} \tag{27c}$$

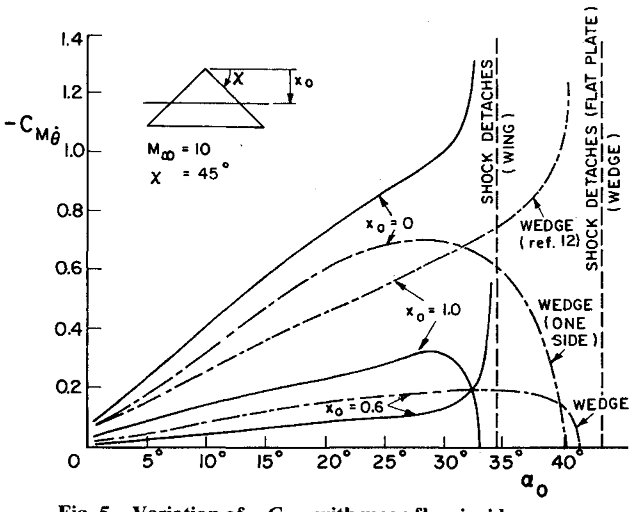

Fig. 5 Variation of $-C_{M_{\dot\theta}}$ with mean flow incidence α_0.

Consequently, the damping boundary can be determined simply by requiring $C_{M_{\dot\theta}}=0$ in Eq. (27). It is observed that the resulting moment expressions [Eq. (27] are rather involved, unlike the simple expression obtained by Hui and Hemdan.[13] We believe this complicated algebraic result obtained above is a consequence of our problem being basically a conically mixed-type problem, whereas in Hui and Hemdan's thin shock layer approach, the problem is a conically hyperbolic one, hence it renders a much simpler result.

VI. Results and Discussion

To demonstrate the present method, the following numerical results are presented to exhibit the dependence of the moments on various flow parameters and geometrical parameters. Figure 2 is a typical diagram which shows the in-phase and the damping moment versus pitch axis location x_0 for a delta wing and a one-side wedge§ at the same Mach number ($M_\infty=4.0$) and the same incidence ($\alpha_\circ=15°$). In Fig. 3, the variation of the damping moment is plotted against the sweepback angle χ. It is seen that, for a given x_0, the damping moment varies little with χ except near the shock detachment region. Figure 4 is a study of the effect of freestream Mach numbers on the damping moment. Also it is interesting to observe that the damping derivatives are rather insensitive to the higher Mach numbers (starting from $M_\infty=6.0$, for example) consistent with the principle of Mach number independence in hypersonic flow. Again, drastic behavior of these trends is expected when they are near the shock detachment zone. Furthermore, note that when $\chi\to0$, $C_{M_{\dot\theta}}$ for a delta wing does not approach to that defined for a one-side wedge, but differs roughly by a ratio of 2/3. An explanation for this is given as follows. The damping derivative $C_{M_{\dot\theta}}$. for a wedge in general is defined by a single integral whereas for a delta wing, by one more integral in the spanwise direction; hence, the difference in the $C_{M_{\dot\theta}}$ values is only introduced by the definition. For example, if we only account for the pressure in the outer region, the ratio between a one-side wedge and a delta wing reads

$$\left(\frac{C_{M_{\dot\theta}\,\mathrm{wedge}}}{C_{M_{\dot\theta}\,\mathrm{delta}}}\right)_{\chi\to0}=\frac{2}{3}$$

In Fig. 5, comparatively large variation of $C_{M_{\dot\theta}}$ shows its strong dependence on the incidence α_0, in contrast to the slight dependence on χ and M_∞. Finally, Fig. 6 shows the neutral damping boundaries for delta wings and wedges[12] (one-side and two-side), i.e. $C_{M_{\dot\theta}}=0$ [see Eq. (27]. It is observed that the sweepback effect ($\chi=10°$) is to enlarge the stable zone in the high Mach number ranges. According to our calculation for other cases of sweepback angle, the critical pivot position is almost invariant, occurring aroung $x_0=1/2$. On the other hand, such position for a one-side wedge occurs at $x_0=1/3$. The reason for such a position shift is again due to the earlier explanation.

During the course of the present study, we have gone through a long and painstaking search of the previous literature concerning the available data, experiment or theory, suitable for comparison with the present theory. Unfortunately, it was difficult to find any theoretical result based on the linearized theory (e.g. Ref. 26) for reasonable comparisons, because usually most previous cases are only applicable for $\alpha_0=0$ and our theory is restricted to cases where $\alpha_0\neq0$. To the best of our knowledge, the experimental data relevant for comparison with present theory are virtually nonexistent. Hence, we have to await new future results for a meaningful comparison.

§For a one-sided wedge, we really mean a flat plate relating to a delta wing case of $\chi\to0$, whose pitching axis lies on the plane of the mean surface. For a wedge (or called two-side wedge), of course, the pitching axis is located along the line of symmetry, which bisects the wedge angle.

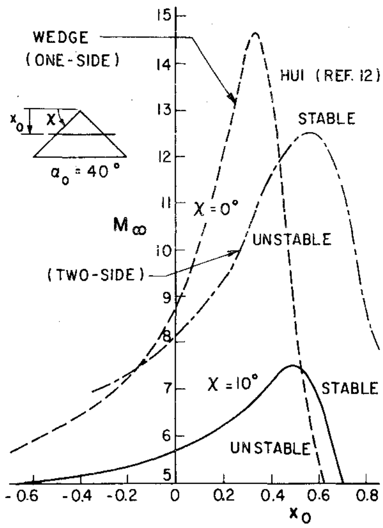

Fig. 6 Neutral damping boundaries for delta wings and wedges.

VII. Conclusions

In essence, the present problem in the crossflow plane is characterized by the mixed-type equations. Thus it can be viewed as another type of transonic flow problem in which the equations solved are linear and the physical regions of interest are bounded in a finite domain. Several interesting features of the out-of-phase flow have been found. First, the out-of-phase flow is proved to be 'quasiconical,' thus allowing the problem to be formulated in terms of pressure alone as a dependent variable. Second, in the outer region, exact solutions for pressure, density, and velocities are found as linear functions of the space coordinates, representing parallel surfaces of isobars (and iso-velocities). The out-of-phase shock shape, consequently, turns out to be a quadratic function of the space coordinates. Third, in the inner region, the method of spanwise integration is generalized to reduce the problem to an ordinary-differential-equation formulation. Again, closed form solutions are found. Damping derivatives are then obtained in algebraic forms.

Hence, a theory has been developed for oscillating delta wings with attached shock waves. It provides damping derivatives as a function of a number of flow and geometrical parameters. In particular, the dampign moment is found to be more sensitive to the mean incidence than to the sweepback angles and freestream Mach numbers.

Finally, we conclude that the present work is a first attempt in the unsteady wing theory in which the shock wave effect, hence the rotationality, is properly accounted for and the Mach number range unified. To obtain an accurate result, however, the given flow condition for application is restricted to one in which the interaction between unsteady wedge flow and the steady mean flow is relatively small in comparison with the unsteady flow contribution strictly from the outer region. Nonetheless, this restriction can be removed following a successive approximation scheme proposed by Liu.[20] It would be interesting to pursue the problem along this line for a further investigation. Also, we believe that the present theory can be extended to the studies of unsteady rolling and yawing problems of the same configuration, without much difficulty.

Acknowledgments

The authors would like to thank H.P. Chou of the University of Southampton for checking the complicated algebra in this work; they also would like to acknowledge G.M. Lilley and R.A. East of the same institution for their prolonged interest and helpful comments concerning the problem. This work was carried out and completed at the University of Southampton with the support of Procurement Executive, British Ministry of Defence. Partial support by a grant from the National Research Council of Canada is also acknowledged.

References

[1]Townend, L.H., "Some Design Aspects of Space Shuttle Orbitors," RAE TR 70139, 1970.

[2]Orlik-Ruckemann, "Dynamic Stability Testing of Aircraft – Needs versus Capabilities," *Progress in the Aerospace Sciences*, Vol. 16, Academic Press, N.Y., 1975, pp. 431-447.

[3]Fowell, L.R., "Exact and Approximate Solutions for the Supersonic Delta Wing," *Journal of the Aeronautical Sciences*, Vol. 8, 1956, pp. 709-720.

[4]Babaev, D.A., "Numerical Solution of the Problem of Supersonic Flow Past the Lower Surface of a Delta Wing," *AIAA Journal*, Vol. 1, Sept. 1963, pp. 2224-2231.

[5]Voskresenskii, G.P., "Numerical Solution of the Problem of a Supersonic Gas Flow Past an Arbitrary Surface of a Delta Wing in the Compression Region," *Izv. Akad. Nauk SSR*, Mekh. Zhidk., Gaza, No. 4, 1968, 134-142.

[6]South, J.C. and Klunker, E.B., "Methods for Calculating Non-Linear Conical Flows," NASA SP 228, Oct. 1969, pp. 131-156.

[7]Kutler, P. and Lomax, H., "Shock Capturing Finite-Difference Approach to Supersonic Flows," *Journal of Spacecraft and Rockets*, Vol. 8, Dec. 1971, pp. 1175-1182.

[8]Malmuth, N., "Hypersonic Flow Over a Delata Wing of Moderate Aspect Ratio," *AIAA Journal*, Vol. 4, March 1966, pp. 555-556.

[9]Malmuth, N., "A New Area Rule for Hypersonic and Supersonic Wing-Bodies," *AIAA Journal*, Vol. 9, Dec., 1971, pp. 2460-2462.

[10]Hui, W.H., "Supersonic and Hypersonic Flow with Attached Shock Waves Over Delta Wings," *Proceedings of the Royal Society*, London, A325, 1971, pp. 251-268.

[11]Hui, W.H., "Unified Area Rule for Hypersonic and Supersonic Wing-Bodies," *AIAA Journal*, Vol. 10, July 1972, pp. 961-962.

[12]Hui, W.H., "Stability of Oscillating Wedges and Caret Wings in Hypersonic and Supersonic Flows," *AIAA Journal*, Vol. 7, Aug. 1969, pp. 1524-1530.

[13]Hui, W.H. and Hemdan, H.T., "Unsteady Hypersonic Flow over Delta Wings with Detached Shcok Waves," *AIAA Journal*, Vol. 14, April 1976, pp. 505-511.

[14]Garrick I.E. and Rubinow, S.I., "Theoretical Study of Air Forces on an Oscillating or Steady Thin Wing in a Supersonic Main Stream," NACA Rept. 872, 1947.

[15]Miles, J.W., "Potential Theory of Unsteady Supersonic Flow," Cambridge Monographs on Mechanics and Applied Mathematics, 1959.

[16]Landahl, M.T., "Unsteady Flow Around Thin Wings at High Mach Numbers," *Journal of the Aeronautical Sciences*, Vol. 24, 1957, pp. 33-38.

[17]Van Dyke, M.D., "Supersonic Flow Past Oscillating Airfoils Inclusing Non-Linear Theickness Effects," NACA Rept. 1183, 1954.

[18]Ashley, H. and Zartarian, G., "Piston Theory-A New Aerodynamic Tool for the Aeroelastician," *Journal of the Aeronautical Sciences*, Vol. 23, Dec. 1956, pp. 1109-1118.

[19]Miles, J.W., "Newtonian Flow Over a Stationary Body in Unsteady Flow," *AIAA Journal*, Vol. 4, Jan., 1966, pp. 38-40.

[20]Liu, D.D., "A Unified Unsteady Flow Theory of Delta Wings with Attached Shcok Waves," Ph.D. Thesis,The University of Southampton, Aug. 1974.

[21]Malmuth, N., "Pressure Fields Over Hypersonic Wing-Bodies at Moderate Incidence," *Journal of Fluid Mechanics*, Vol. 59, Part 5, 1973, pp. 673-691.

[22]Lomax, H. and Heaslet, M.A., "Generalized Conical-Flow Fields in Supersonic Wing Theory," NACA TN 2497, Sept. 1951.

[23]Malvestuto, F.S. and Margolis, K., "Theoretical Stability Derivatives of Their Sweptback Wings Tapered to a Point with Sweptback or Sweptforward Trailing Edges for a Limited Range of Supersonic Speeds," NACA TN 1761, Jan. 1949.

[24]Miles, J.W., "On Nonsteady Motion of Delta Wings," *Journal of the Aeronautical Sciences*, Vol. 16, Sept. 1949, pp. 568-569.

[25]Froehlich, J.E., "Nonstationary Motion of Purely Supersonic Wings," *Journal of the Aeronautical Sciences*, Vol. 18, 1951, pp. 298-310.

[26]Brown, C.E. and Adams, M.C., "Damping In-pitch and Roll of Triangular Wings at Supersonic Speed," NACA TN 1566, 1948.